A COUNTEREXAMPLE IN THE THEORIES OF COMPACTNESS AND OF METRIZATION [1]

BY

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The fact that the Baire category theorem held for both compact Hausdorff spaces and complete metric spaces led to a search for a natural class of spaces encompassing both of these classes and satisfying the theorem. The first successful attempt was due to ČECH [5] – the so-called *Čech complete* spaces. There have been numerous additional attempts since then. In addition to Čech completeness, we shall only deal with two concepts closely related to each other, *basis-compactness* [10] and *co-compactness* [1]. A comprehensive survey [2] of these and many more completeness properties by J. M. AARTS and D. J. LUTZER is in preparation. I am indebted to them for many stimulating thoughts on the subject.

We shall present an example to show that neither basis-compactness nor cocompactness is implied by Čech completeness. The example has a number of other interesting properties – it is a metacompact locally metrizable Moore space which is not metrizable, but may, consistently with the axioms of set theory, be assumed normal. The example's significance in metrization theory is dealt with in [9]. Here we will confine ourselves to completeness questions.

DEFINITION. A space (we assume all spaces completely regular) is *Čech complete* if it is a G_δ in its Stone-Čech compactification.

DEFINITION. A collection of sets is *centered* if every finite subcollection has non-empty intersection. A space is *basis-compact* if it has a base $\mathscr{B}$ (called a *compact base*) such that if $\mathscr{F} \subset \mathscr{B}$ is centered then $\bigcap \{\overline{F} : F \in \mathscr{F}\} \neq \emptyset$. A space is *cocompact* if there is a collection of closed sets (called a *co-compact closed base*) such that if $p \in U$ open, then there is $B \in \mathscr{B}$, $p \in \operatorname{int} B$, $B \subset U$, and every centered subcollection of $\mathscr{B}$ has non-empty intersection.

All three of these properties are equivalent to the usual topological completeness in metric spaces, and imply the Baire theorem. The reasons they are interesting can be found in the references.

It is known that the product of $2^{\aleph_0}$ copies of the real line is basis-compact and cocompact, but not Čech complete. (See [10] and [1]). The point is that the first two properties are productive, while the third is

[1]) I should like to dedicate this note to the late Johannes de Groot, who was indeed a gentleman and a scholar.

inherited by closed sets. The rationals are obviously not Čech complete, but can be embedded in this product of lines as a closed subset [7].

The problems of whether Čech complete spaces are basis-compact or cocompact are raised in [10] and [1] respectively, and are originally due to J. de Groot. To construct the counterexample we start with a familiar space and split some of its points. The origins of this idea can be found (well-hidden!) in a paper of Bing [4].

Let Y be an uncountable subset of the real line. Let D consist of all those points in the plane above the line having both coordinates rational. Define a topology on $X_0 = D \cup Y$ as follows. Each $\{d\}$, $d \in D$ is open. For each $y \in Y$, and each $n \in \omega$, pick $d(y, n) \in D$ such that $1/n+1 < \varrho(y, d(y, n)) < 1/n$, where ϱ is the usual metric on the plane. Thus $\{d(y, n)\}_{n \in \omega}$ converges monotonically to y. Let $M(y, n) = \{y\} \cup \{d(y, m) : m \geqslant n\}$. We take $\{M(y, n)\}_{n \in \omega}$ as a base for $y \in Y$.

It is easy to see that Y is a closed discrete subspace of X_0, D is dense in X_0, X_0 is Hausdorff, and each basic open set is compact. We now modify X_0 to construct a new space X as follows. The points of X will consist of the points of Y and the members of $D^{\#} = \{\langle d, F\rangle : d \in D,\ F$ a finite subset of $Y\}$. We write d_F for $\langle d, F\rangle$. Thus each $d \in D$ splits into as many points as there are finite subsets of Y. The topology for $X = Y \cup D^{\#}$ is defined as follows: first, each $\{d_F\}$, $d_F \in D^{\#}$, is open. Second, define a base at $y \in Y$ in X to be $\{N(y, n)\}_{n \in \omega}$, where $N(y, n) = \{y\} \cup \{d_F : y \in F$ and $d \in M(y, n)\}$.

Let $\mathscr{J}_n = \{\{d_F\} : d_F \in D^{\#}\} \cup \{N(y, n) : y \in Y\}$. Then observe that $\mathscr{J} = \bigcup_{n \in \omega} \mathscr{J}_n$ is a σ-point-finite base for X, because for each n, every d_F is in at most one more than the cardinality of F members of $\mathscr{J}_n$, while each $y \in Y$ is in only one member of $\mathscr{J}_n$. (In fact $\{\mathscr{J}_n\}_{n \in \omega}$ is a *development* for X, see [9]).

The crucial point of the construction is that if $y_1, \ldots, y_i \in Y$ and $n_1, \ldots, n_i \in \omega$, then $N(y_1, n_1) \cap \ldots \cap N(y_i, n_i) = \emptyset$ if and only if $M(y_1, n_1) \cap \ldots \cap M(y_i, n_i) = \emptyset$. For $d_{\{y_1, \ldots, y_i\}} \in N(y_1, n_1) \cap \ldots \cap N(y_i, n_i)$ if and only if $d \in M(y_1, n_1) \cap \ldots \cap M(y_i, n_i)$. It follows easily that since X_0 is Hausdorff, so is X. (It also follows that X is not *collectionwise normal,* since X_0 isn't, and that X is normal if X_0 is. If Y is chosen to be a subset of the reals such that every subset of Y is a relative F_σ, then the same proof as for Example E of [3] shows that X_0 is normal. The existence of such an uncountable Y is consistent with the axioms of set theory, Silver [8, section 2.3].) It also is easy to see that each basic open set is closed, so X is completely regular.

To show that X is Čech complete, we use an equivalent definition.

Lemma [6]. A space is Čech complete if and only if it possesses a sequence $\{\mathscr{H}_n\}_{n \in \omega}$ of open covers (called a *complete sequence*), such that whenever $\mathscr{K}$ is a centered collection of closed sets such that $(\forall n)(\exists H \in \mathscr{H}_n)(\exists K_n \in \mathscr{K})(K_n \subset H)$, then $\bigcap \mathscr{K} \neq \emptyset$.

For example, note that $\{\mathscr{H}_n\}_{n\in\omega}$ is a complete sequence of open covers for X_0, where $\mathscr{H}_n=\{\{d\}: d\in D\}\cup\{M(y, n): y\in Y\}$. We claim that $\{\mathscr{J}_n\}_{n\in\omega}$ is a complete sequence of covers for X. Let $\mathscr{K}$ be a collection of closed subsets of X which is centered, $K_n\subset G_n\in\mathscr{J}_n$. If any G_n is a singleton, it follows that $\bigcap\mathscr{K}=G_n$, so without loss of generality, assume $G_n=N(y_n, n)$. $\{N(y_n, n)\}_{n\in\omega}$ is centered since $\{K_n\}_{n\in\omega}$ is. Therefore so is $\{M(y_n, n)\}_{n\in\omega}$. But each $M(y_n, n)$ is compact, so $M=\bigcap_{n\in\omega}M(y_n, n)\neq\emptyset$.

Considering the usual metric in the plane, we see that $M\subset Y$. In fact M must consist of a single $y\in Y$, for if $y, y'\in M$, then there exist arbitrarily large n such that $y=y_n$, and arbitrarily large m such that $y'=y_m$. But this contradicts X_0 Hausdorff. We conclude that all but finitely many y_n are equal to y. But then $\bigcap_{n\in\omega}N(y_n, n)=\{y\}$. It follows that $\bigcap_{n\in\omega}K_n=\{y\}$. Certainly $\bigcap_{n\in\omega}K_n\subset\{y\}$, since $K_n\subset N(y_n, n)$. If $y\notin$ some K_n, then since K_n is closed, there is an m such that for every $k\geqslant m$, $N(y, k)\cap K_n=\emptyset$. But, taking k sufficiently large, $y_k=y$, $N(y_k, k)\supset K_k$, and $K_k\cap K_n\neq\emptyset$. We are done, for if $\bigcap_{n\in\omega}K_n=\{y\}$, so does $\bigcap\mathscr{K}$.

It remains to show that X is neither cocompact nor basis-compact. The proofs are similar. Suppose $\mathscr{B}$ is a cocompact closed base for X. For each $y\in Y$ and each $n\in\omega$, pick $B(y, n)\in\mathscr{B}$ such that $y\in\operatorname{int}B(y, n)\subset$ $\subset B(y, n)\subset N(y, n)$. Then $\mathscr{B}'=\{\{p\}: p\in D^*\}\cup\{B(y, n): y\in Y, n\in\omega\}$ is a cocompact closed base. Furthermore, $\mathscr{B}'$ is σ-point-finite since $\mathscr{J}$ is. Since $\mathscr{B}'$ is cocompact, no uncountable subcollection of $\mathscr{B}'$ can be centered, for this would contradict $\mathscr{B}'$ σ-point-finite. Alternatively, suppose $\mathscr{B}$ is a compact base for X. As before, pick $B(y, n)\in\mathscr{B}$, $y\in\overline{B(y, n)}\subset N(y, n)$. $\mathscr{B}'$ defined as above is then a compact base. $\overline{\mathscr{B}'}=\{\overline{B}: B\in\mathscr{B}'\}$ is σ-point-finite since $\mathscr{J}$ is. Since $\mathscr{B}'$ is compact, it follows that no uncountable subcollection of $\mathscr{B}'$ can be centered, for this would contradict $\overline{\mathscr{B}'}$ σ-point-finite.

Whichever $\mathscr{B}'$ is in question, for each $y\in Y$, pick $B_y\in\mathscr{B}'$, $y\in B_y\subset$ $\subset N(y, 0)$. Then $y\neq y'$ implies $B_y\neq B_{y'}$. Hence $\{B_y\}_{y\in Y}$ is uncountable. Then for each y pick $N(y, n)$ such that $y\in N(y, n)\subset B_y$. Consider $\{M(y, n): y\in Y\}$. This is an uncountable collection of open sets in a separable space, so there is an uncountable $Y'\subset Y$, such that $\{M(y, n): y\in Y'\}$ is centered. But then $\{N(y, n): y\in Y'\}$ is centered, and hence so is $\{B_y\}_{y\in Y'}$, contradiction.

Although X is not basis-compact or cocompact, it has these properties locally. The point is that X can be shown to be locally metrizable. Čech completeness is inherited by open sets, so X is locally completely metrizable, hence locally basis-compact and cocompact. Aarts and Lutzer have called to my attention that it follows that neither basis-compactness nor cocompactness is preserved by open continuous maps. For X is the open continuous image of the disjoint sum of its open completely metrizable subspaces. This sum is completely metrizable and hence basis-compact and cocompact.

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